

On Finite Metahamiltonian p -Groups *

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Abstract

A group is called metahamiltonian if all non-abelian subgroups of it are normal. This concept is a natural generalization of Hamiltonian groups. In this paper, the properties of finite metahamiltonian p -groups are investigated.

Keywords Dedekindian groups, metahamiltonian groups, \mathcal{A}_2 -groups

2000 Mathematics subject classification: 20D15.

1 Introduction

A group is called Dedekindian if every subgroup of it is normal. In 1897, Dedekind classified finite Dedekindian groups in [6]. In 1933, Baer classified infinite Dedekindian groups in [1]. A non-abelian Dedekindian group is also called Hamiltonian.

A non-abelian group is called metahamiltonian if all non-abelian subgroups of it are normal. This concept is a natural generalization of Hamiltonian groups. In the 1960's and 70's, many scholars researched metahamilton groups. Romalis and Sesekin [16, 17, 18] investigated some properties on infinite metahamiltonian groups, and Nagrebeckii [11, 12, 13] studied finite metahamiltonian groups. Nagrebeckii [12] proved the following theorem:

Theorem 1.1. *Suppose that G is a finite non-nilpotent group. Then G is metahamiltonian if and only if $G = SZ(G)$ where S is one of the following groups:*

- (1) $P \rtimes Q$, where P is an elementary p -group, Q is cyclic and $(p, |Q|) = 1$;
- (2) $Q_8 \rtimes Q$, where Q is cyclic and $(|Q|, 2) = 1$;
- (3) $P \rtimes Q$, where $|P| = p^3, p \geq 5$, Q is cyclic and $(p, |Q|) = 1$.

In [12], more detailed information on S is given. Since a nilpotent group is the direct product of its Sylow subgroups, by the above theorem, to study finite metahamiltonian groups, we only need consider finite metahamiltonian p -groups, which is more complex than the situation of non-nilpotent.

*This work was supported by NSFC (no. 11371232), by NSF of Shanxi Province (no. 2013011001-1) and Shanxi Scholarship Council of China (No. [2011]8-059).

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Metahamiltonian p -groups contain many important classes of p -groups. For example, finite p -groups all of whose subgroups of index p^2 are abelian, are metahamiltonian. All such groups are determined, see [3, 4, 7, 9, 19, 21] for the classification. Another example is finite p -groups all of whose non-normal subgroups are cyclic. See [14]. The study of metahamilton p -groups is an old problem and many scholar consider it important. In this paper, the properties of finite metahamiltonian p -groups are investigated. These properties are useful in the classification of metahamilton p -groups [8].

2 Preliminaries

Let G be a finite group. G is said to be *minimal non-abelian*, if G is non-abelian, but every proper subgroup of G is abelian. A finite p -group G is called an \mathcal{A}_t -group if every subgroup of index p^t of G is abelian, but there is at least one non-abelian subgroup of index p^{t-1} . So \mathcal{A}_1 -groups are just the minimal non-abelian p -groups.

Let G be a finite p -group. We define $\Lambda_1(G) = \{a \in G \mid a^p = 1\}$, $V_1(G) = \{a^p \mid a \in G\}$, $\Omega_1(G) = \langle \Lambda_1(G) \rangle = \langle a \in G \mid a^p = 1 \rangle$, and $U_1(G) = \langle V_1(G) \rangle = \langle a^p \mid a \in G \rangle$; G is called *p -abelian* if $(ab)^p = a^p b^p$ for all $a, b \in G$. We use $c(G)$ and $d(G)$ to denote the nilpotency class and minimal number of generators, respectively.

We use $M_p(m, n)$ to denote groups $\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle$, where $m \geq 2$, and use $M_p(m, n, 1)$ to denote groups $\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, where $m + n \geq 3$ for $p = 2$ and $m \geq n$. We can give a presentation of minimal non-abelian p -groups as follows:

Theorem 2.1. ([15])(Rédei) *Let G be a minimal non-abelian p -group. Then G is Q_8 , $M_p(m, n)$, or $M_p(m, n, 1)$.*

We use C_n and C_n^m to denote the cyclic group and the direct product of m cyclic groups of order n , respectively; and use $H * K$ to denote a central product of H and K . For undefined notation and terminology the reader is referred to [10].

We have the following information about minimal non-abelian p -groups.

Theorem 2.2. ([20, Lemma 2.2]) *Let G be a finite p -group. Then the following conditions are equivalent:*

- (1) G is an inner abelian p -group;
- (2) $d(G) = 2$ and $|G'| = p$;
- (3) $d(G) = 2$ and $Z(G) = \Phi(G)$.

Lemma 2.3. ([2, p₁₃₆, Proposition 10.28]) *A non-abelian p -group is generated by minimal non-abelian subgroups.*

Many scholars studied and classified \mathcal{A}_2 -groups, see, for example [3, 4, 7, 9, 19, 21]. We have following Lemma.

Lemma 2.4. ([21]) *Suppose that G is an \mathcal{A}_2 -group. Then G is one of the following groups:*

- (I) $d(G) = 2$ and G has an abelian maximal subgroup.
- (1) $\langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^{-1} \rangle$, where $m \geq 1$;
 - (2) $\langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^3 \rangle$, where $m \geq 1$;
 - (3) $\langle a, b \mid a^8 = 1, b^{2^m} = a^4, a^b = a^{-1} \rangle$, where $m \geq 1$;
 - (4) $\langle a_1, b \mid a_1^p = a_2^p = a_3^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_i, a_j] = 1 \rangle$, where $p \geq 5$ for $m = 1$, $p \geq 3$ and $1 \leq i, j \leq 3$;
 - (5) $\langle a_1, b \mid a_1^p = a_2^p = b^{p^{m+1}} = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_1, a_2] = 1 \rangle$, where $p \geq 3$;
 - (6) $\langle a_1, b \mid a_1^{p^2} = a_2^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_1, a_2] = 1 \rangle$, where $p \geq 3$ and $\nu = 1$ or a fixed quadratic non-residue modulo p .
 - (7) $\langle a_1, a_2, b \mid a_1^9 = a_2^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [a_2, a_1] = 1 \rangle$.
- (II) $d(G) = 3$, $|G'| = p$ and G has an abelian maximal subgroup.
- (8) $\langle a, b, x \mid a^4 = x^2 = 1, b^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 \times C_2$;
 - (9) $\langle a, b, x \mid a^{p^{n+1}} = b^{p^m} = x^p = 1, [a, b] = a^{p^n}, [x, a] = [x, b] = 1 \rangle \cong M_p(n+1, m) \times C_p$;
 - (10) $\langle a, b, c, x \mid a^{p^n} = b^{p^m} = c^p = x^p = 1, [a, b] = c, [c, a] = [c, b] = [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) \times C_p$, where $n \geq m$, and $n \geq 2$ if $p = 2$;
 - (11) $\langle a, b, x \mid a^4 = 1, b^2 = x^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 * C_4$;
 - (12) $\langle a, b, x \mid a^{p^n} = b^{p^m} = x^{p^2} = 1, [a, b] = x^p, [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) * C_{p^2}$, where $n \geq 2$ if $p = 2$ and $n \geq m$.
- (III) $d(G) = 3$, $|G'| = p^2$ and G has an abelian maximal subgroup.
- (13) $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2 b^2, [a, b] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$;
 - (14) $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^p = 1, [a, b] = a^{p^{m-1}}, [d, a] = b^p, [d, b] = 1 \rangle$, where $m \geq 3$ if $p = 2$;
 - (15) $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}, [d, b] = 1 \rangle$, where $(j, p) = 1$, $p > 2$, j is a fixed quadratic non-residue modulo p , and $-4j$ is a quadratic non-residue modulo p ;
 - (16) $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp} d^p, [d, b] = 1 \rangle$, where if p is odd, then $4j = 1 - \rho^{2r+1}$ with $1 \leq r \leq \frac{p-1}{2}$ and ρ the smallest positive integer which is a primitive root (mod p); if $p = 2$, then $j = 1$.
- (IV) $d(G) = 2$ and G has no abelian maximal subgroup.

- (17) $\langle a, b \mid a^{p^{r+2}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle$, where $r \geq 2$ for $p = 2$, $r \geq 1$ for $p \geq 3$, $t \geq 0$, $0 \leq s \leq 2$ and $r + s \geq 2$;
- (18) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p \rangle$, where $p \geq 5$, ν is a fixed square non-residue modulo p ;
- (19) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^{-p}b^{-lp}, [c, b] = a^{-p} \rangle$, where $p \geq 5$, $4l = \rho^{2r+1} - 1$, $r = 1, 2, \dots, \frac{1}{2}(p-1)$, ρ is the smallest positive integer which is a primitive root modulo p ;
- (20) $\langle a, b \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^3 \rangle$;
- (21) $\langle a, b \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^{-3} \rangle$.

(V) $d(G) = 3$ and G has no abelian maximal subgroup.

- (22) $\langle a, b, d \mid a^4 = b^4 = d^4 = 1, [a, b] = d^2, [d, a] = b^2d^2, [d, b] = a^2b^2, [a^2, b] = [b^2, a] = 1 \rangle$.

Analyzing the group list in Lemma 2.4, we have following lemma.

Lemma 2.5. *Suppose that G is an \mathcal{A}_2 -group with order p^n .*

- (1) $d(G) \leq 3$ and $c(G) \leq 3$;
- (2) If $d(G) = 2$ and $\exp(G') = p$, then $c(G) = 3$.
- (3) If $c(G) > 2$ and $\exp(G') = p$, then $d(G) = 2$ and p is odd.

Theorem 2.6. ([10, Satz 6.5] *If $[x, y, y] = 1$ for all $x, y \in G$, then G is nilpotent and $c(G) \leq 3$. In addition, if G has no element of order 3, then $c(G) \leq 2$.*

A finite p -group G is called *metacyclic* if it has a cyclic normal subgroup N such that G/N is also cyclic.

Lemma 2.7. ([5]) *Suppose that G is a finite p -group. Then G is metacyclic if and only if $G/\Phi(G')G_3$ is metacyclic.*

3 Properties of finite metahamiltonian p -groups

Theorem 3.1. *Let G be a finite metahamiltonian p -group. Then sections of G are all metahamiltonian.*

Proof It is straight forward. □

Theorem 3.2. *Let G be a finite p -group. Then G is metahamiltonian if and only if every minimal non-abelian subgroup is normal in G .*

Proof If G is metahamiltonian, then, by the definition of metahamiltonian, every minimal non-abelian subgroup is normal in G . On the other hand, if every minimal non-abelian subgroup is normal in G , then, by Lemma 2.3, every non-abelian subgroup is normal in G . □

Theorem 3.3. *Let G be a finite metahamiltonian p -group. Then, for all $x \in G$, $\langle x \rangle^G$ is abelian or minimal non-abelian.*

Proof Suppose that $\langle x \rangle^G$ is not abelian. Then there exists $g \in G$ such that $[x, x^g] \neq 1$. Let $K = \langle x, x^g \rangle$. Then K is normal in G since G is metahamiltonian. Hence $K = \langle x \rangle^G$. Let $y = x^g$ and $L = \langle x, x^y \rangle = \langle x, [x, y] \rangle$. Then $L < K$ and hence L is not normal in G . It follows that L is abelian. That is, $[x, y, x] = 1$. Since $\langle y \rangle^G = \langle x^g \rangle^G = \langle x \rangle^G$, similarly we have $[x, y, y] = 1$. Hence $c(K) = 2$.

Let $S = \langle x, y^p \rangle$. Then $S < K$ and hence S is not normal in G . It follows that S is abelian and hence $[x, y^p] = 1$. Since $c(K) = 2$, we get $[x, y]^p = 1$. Thus $K' = \langle [x, y] \rangle$ is of order p . By Theorem 2.2, K is minimal non-abelian. \square

Theorem 3.4. *Let G be a metahamiltonian p -group. Then $c(G) \leq 3$. In particular, G is metabelian.*

Proof By Theorem 3.3, for all $x \in G$, $K = \langle x \rangle^G$ is abelian or minimal non-abelian. Then $K' = 1$ or $|K'| = p$. Since $K' \trianglelefteq G$, we get $K' \leq Z(G)$. Let $\bar{G} = G/Z(G)$. Then, for all $\bar{x} \in \bar{G}$, $\langle \bar{x} \rangle^{\bar{G}}$ is abelian. Hence \bar{G} satisfies the 2-Engel condition. By Theorem 2.6, $c(\bar{G}) = 2$ for $p \neq 3$ and $c(\bar{G}) \leq 3$ for $p = 3$. It follows that $c(G) \leq 3$ for $p \neq 3$ and $c(G) \leq 4$ for $p = 3$.

We claim that $c(G) \leq 3$. If not, then $p = 3$ by the above argument. Let G be a counterexample with minimal order. By Theorem 3.1, $c(G) = 4$, $|G_4| = p$ and the nilpotency class of every proper section of G is at most 3. Hence we may assume that $G_4 = \langle [a, b, c, d] \rangle$, where $a, b, c, d \in G \setminus \Phi(G)$. Let $x = [a, b, c]$. Then $N = \langle x, d \rangle$ is minimal non-abelian by Theorem 2.2. By hypothesis, every subgroup which contains N is normal in G . It follows that G/N is Dedekindian. Since $p = 3$, G/N is abelian. It follows that $G' \leq N$. Since $d \notin \Phi(G)$, we have $N \cap \Phi(G) < N$ and hence $G' \leq N \cap \Phi(G) < N$. It follows that G' is abelian. Then $[[c, d], [a, b]] = 1$. Since $[a, b] \in G' < N$ and $d \in N$, $[d, [a, b]] \in N' \leq Z(G)$. It follows that $[d, [a, b], c] = 1$. By Witt's formula, we have $[[a, b], c, d] = 1$, a contradiction. \square

Theorem 3.5. *Let G be a finite p -group. G is metahamiltonian if and only if G' is contained in every non-abelian subgroup of G .*

Proof If G' is contained in every non-abelian subgroup of G , then every non-abelian subgroup of G is normal in G . Hence sufficiency holds. In the following we prove the necessity.

Let G be a counterexample with minimal order. Then G is metahamiltonian and there exists a minimal non-abelian subgroup $N = \langle a, b \rangle$ such that $G' \not\leq N$. Since G is metahamiltonian, subgroups containing N are normal in G . Hence G/N is Hamiltonian.

By the minimality of G , $G/N \cong Q_8$. Let $G/N = \langle xN, yN \rangle$ and $H = \langle x, y \rangle$. Then $G = HN$, $H/(H \cap N) \cong Q_8$, $z := [x, y] \notin N$, $H \cap N \leq \Phi(H)$ and $H \cap N =$

$\langle x^4, x^2y^2, x^2[x, y] \rangle^H$. Since $z \in \langle x \rangle^H$, it follows from Theorem 3.3 that $\langle z, x \rangle$ is abelian or minimal non-abelian. Hence $[z, x^2] = [z, x]^2 = 1$. The same reason gives that $[z, y^2] = [z, y]^2 = 1$ and hence $\exp(H_3) \leq 2$. Since $\Phi(H) = \langle x^2, y^2, H' \rangle$ and H' is abelian (by Theorem 3.4), we have $[\Phi(H), z] = 1$. In particular, $[H \cap N, z] = 1$. In the following, we deduce a contradiction on five cases:

Case 1. $H \cap N = N$.

In this case, $[N, z] = 1$. Let $M = \langle za, b \rangle$. Then Theorem 2.2 gives that M is minimal non-abelian, and hence G/M is also Dedekindian. Since $z \notin M$, G/M is not abelian. By the minimality of G , $H/M = G/M \cong Q_8$. It follows that $M = \langle x^4, x^2y^2, x^2[x, y] \rangle^H = N = \langle a, b \rangle$, a contradiction.

Case 2. $H \cap N < N$ and $H \cap N \not\leq \Phi(N)$.

In this cases, $H \cap N$ contains a generator of N . Without losing generality, we assume that $a \in H \cap N$ and $b \notin H \cap N$. Then $[z, a] = 1$. Since $H \cap N$ is abelian, we have $[x^2y^2, x^2[x, y]] = 1$, and hence $[x^2, y^2] = 1$. By calculation, we have $[x^2, y^2] = [x^2, y]^2 = [x, y]^4 = z^4$. If $z^2 \neq 1$, then $\langle z^2 \rangle = \Omega_1(H')$ is a minimal normal subgroup of G . Hence we have $z^2 \in Z(G)$. Particularly, $[z, b]^2 = [z^2, b] = 1$.

Subcase 2.1. $[z, b] \neq [a, b]$.

Let $M = \langle za, b \rangle$. By Theorem 2.2, M is minimal non-abelian and hence G/M is also Dedekindian. Since $z \notin M$, G/M is not abelian. By the minimality of G , we have $G/M = HM \cong H/(H \cap M) \cong Q_8$. It follows that $H \cap M = \langle x^4, x^2y^2, x^2[x, y] \rangle^H = H \cap N$, and hence $a \in H \cap N = H \cap M \leq M$. Thus $z = (za)a^{-1} \in M$, a contradiction.

Subcase 2.2. $[z, b] = [a, b]$.

Let $L = \langle z, b \rangle \cap N$. Then L is normal in G . Let K be a maximal subgroups of N which contains L such that $K \trianglelefteq G$. Then G/K is of order 2^4 , has two generators, and has a quotient group which is isomorphic to Q_8 . By the classification of groups of order 2^4 , $G/K = \langle xK, yK \rangle := \langle \bar{x}, \bar{y} \rangle \cong M_2(2, 2)$, which has definition relations $\bar{x}^4 = \bar{y}^4 = 1$ and $[\bar{x}, \bar{y}] = \bar{x}^2$. Obviously, $\langle \bar{y} \rangle$ and $\langle \bar{x}\bar{y} \rangle$ are not normal in G/K . It follows that their complete inverse images are also not normal in G , hence are abelian. It follows that $[y, K] = 1$, $[xy, K] = 1$. Thus $[H, K] = 1$, which is contrary to $[z, b] = [a, b] \neq 1$.

Case 3. $H \cap N < \Phi(N)$.

We claim that $H \cap N \neq 1$. Otherwise, $G = H \times N$. Since $N \cong G/H$ is Dedekind, we have $N \cong Q_8$. In this case, $\langle xa, yb \rangle \cong Q_8$ is not normal in G , a contradiction.

We claim that $N' \leq H \cap N$. Otherwise, $G/(H \cap N)$ is also a counterexample, which is contrary to the minimality of G .

Let $\overline{G} = G/(H \cap N)$, $\overline{H} = H/(H \cap N) = \langle \bar{x}, \bar{y} \rangle$ and $\overline{N} = N/(H \cap N) = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$. Then $\overline{G} = \overline{H} \times \overline{N}$ and $\exp(\overline{N}) \geq 4$. Without loss of generality, we may assume that $o(\bar{a}) \geq 4$. Let $\overline{K} = \langle \bar{x}\bar{a} \rangle \times \langle \bar{b} \rangle$. Then \overline{K} is not normal in \overline{G} . It follows that its complete

inverse image is not normal in G . Hence $[xa, b] = 1$. That is, $[x, b] = [a, b]$. The same reason gives that $[y, b] = [a, b]$ and $[xy, b] = [a, b]$, a contradiction.

Case 4. $H \cap N = \Phi(N) = N'$.

In this case, $|N| = 2^3$, $|H| = 2^4$, $|G| = 2^6$ and $G/N' = H/N' \times \langle aN' \rangle \times \langle bN' \rangle$. Since $\langle aN' \rangle$ and $\langle bN' \rangle$ are normal in G/N' , $A := \langle a, N' \rangle$ and $B := \langle b, N' \rangle$, their complete inverse images, are also normal in G . Noting that A and B are of order 4, the NC-Theorem gives that $C_G(A)$ and $C_G(B)$ are maximal in G . Let $K = C_G(A) \cap C_G(B)$. Then $|K| \geq 2^4$. Since $K \cap N = Z(N) = N'$, we have $|KN| = (|K||N|)/|K \cap N| \geq 2^6$, and hence $G = K * N$. Since $KN/N \cong K/K \cap N \cong Q_8$, without loss of generality we may assume that $H = K$. By the classification of groups of order 2^4 , $H = \langle x, y \rangle \cong M_2(2, 2)$, which has definition relations $x^4 = y^4 = 1$, $[x, y] = x^2$ and $N' = H \cap N = \langle x^2 y^2 \rangle$.

Without loss of generality, we may assume that $a \in N$ is of order 4. Then $a^2 = x^2 y^2$. By calculations, we have $[x, ay] = x^2$ and $(ay)^2 = x^2$. It follows that $\langle x, ay \rangle$ is neither abelian nor normal in G , a contradiction.

Case 5. $H \cap N = \Phi(N) \neq N'$.

Let $\overline{G} = G/K$, $\overline{H} = H/K = \langle \bar{x}, \bar{y} \rangle$ and $\overline{N} = N/K = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$, where K is a maximal subgroup of $H \cap N$ such that $K \trianglelefteq G$. By the minimality of G , \overline{G}' is contained in every minimal non-abelian subgroup of \overline{G} . Since $G' \not\leq N$, we have $\overline{G}' \not\leq \overline{N}$ and hence \overline{N} is abelian. Without loss of generality, we may assume that $o(\bar{a}) = 4$. By the classification of groups of order 2^4 , $\overline{H} \cong M_2(2, 2)$, which has definitions $\bar{x}^4 = \bar{y}^4 = 1$, $[\bar{x}, \bar{y}] = \bar{x}^2$, $\bar{a}^2 = \bar{x}^2 \bar{y}^2$, and $\Phi(N)/K = (H \cap N)/K = \langle \bar{x}^2 \bar{y}^2 \rangle$. If $\bar{a} \in Z(\overline{G})$, then $\langle \bar{x}, \bar{a} \bar{y} \rangle$ is neither abelian nor normal in \overline{G} , a contradiction. Hence $\bar{a} \notin Z(\overline{G})$. If $[\bar{a}, \bar{x}] = \bar{1}$, then $[\bar{a}, \bar{y}] = \bar{x}^2 \bar{y}^2$ and hence $\langle \bar{a} \bar{x}, \bar{y} \rangle$ is neither abelian nor normal in \overline{G} , a contradiction. Hence $[\bar{a}, \bar{x}] = \bar{x}^2 \bar{y}^2$. The same reason gives that $[\bar{a} \bar{b}, \bar{x}] = \bar{x}^2 \bar{y}^2$. It follows that $[\bar{b}, \bar{x}] = 1$. If $[\bar{b}, \bar{y}] \neq \bar{1}$, then $[\bar{b}, \bar{y}] = \bar{x}^2 \bar{y}^2$. By calculation, $\langle \bar{x}, \bar{b} \bar{y} \rangle$ is neither abelian nor normal in \overline{G} , a contradiction. Hence $[\bar{b}, \bar{y}] = \bar{1}$.

In this case, it is easy to see that $\langle \bar{x}, \bar{b} \rangle$ and $\langle \bar{a} \bar{x}, \bar{b} \rangle$ are not normal in \overline{G} . It follows that their complete converse images are not normal in G , and hence they are abelian. Thus $[x, b] = 1$ and $[ax, b] = 1$, which is contrary to $[a, b] \neq 1$. \square

Theorem 3.6. *Suppose that G is a finite metahamilton p -group. If $d(G) = 2$ and $\exp(G') > p$, then G is metacyclic.*

Proof Assume that $G = \langle a, b \rangle$ is a counterexample with minimal order. By Lemma 2.7, $\overline{G} := G/\Phi(G')G_3$ is not metacyclic. Since $|\overline{G}'| = p$, \overline{G} is minimal non-abelian. By Theorem 2.1, $\overline{G} \cong M_p(n, m, 1)$. That is, we may assume that $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^n} = \bar{b}^{p^m} = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle$. Since $\langle \bar{a}^p, \bar{b} \rangle$, $\langle \bar{b}^p, \bar{a} \rangle$, $\langle (\bar{a}\bar{b})^p, \bar{a} \rangle$ and $\langle (\bar{a}\bar{b})^p, \bar{b} \rangle$ are not normal in \overline{G} , we have $\langle a^p, b, \Phi(G')G_3 \rangle$, $\langle b^p, a, \Phi(G')G_3 \rangle$, $\langle (ab)^p, a, \Phi(G')G_3 \rangle$ and $\langle (ab)^p, b, \Phi(G')G_3 \rangle$ are not normal in G . Hence they are all abelian. Thus we have

$\Phi(G')G_3 \leq Z(G)$ and

$$[a^p, b] = [b^p, a] = [(ab)^p, a] = [(ab)^p, b] = 1 \quad (*)$$

If $p = 2$, then $(ab)^2 = a^2b^2[a, b]$. By $(*)$, $[a, b] \in Z(G)$. Hence $G' = \langle [a, b] \rangle$. By $(*)$, $[a, b]^2 = [a^2, b] = 1$, which is contrary to $\exp(G') > 2$.

If $p > 2$, then, by calculation, we have $[a, b, a]^p = [a^p, b, a] = 1$ and $[a, b, b]^p = [a^p, b, b] = 1$. It follows that $\exp(G_3) \leq p$. By $(*)$, $[a, b]^p = [a^p, b] = 1$, which is contrary to $\exp(G') > p$. \square

Lemma 3.7. *Suppose that G is a finite metahamiltonian p -group which has elementary abelian derived group. If G is not an \mathcal{A}_2 -group, then \mathcal{A}_2 -subgroups of G have nilpotency class 2.*

Proof Assume the contrary. Then there exists $K < G$ such that $K \in \mathcal{A}_2$, $\exp(K') = p$ and $c(K) \geq 3$. Hence $p > 2$ and K is a group of Type (4)–(7) or (18)–(21) in Lemma 2.4.

Case 1: $K = \langle a_1, b \rangle$ is a group of Type (4)–(7) in Lemma 2.4.

Let $H \leq G$ such that $K < H$. Since $K/Z(K)$ is minimal non-abelian, non-metacyclic and of order p^3 , $H/Z(K)$ is not metacyclic and of order p^4 . If $d(H/Z(K)) = 2$, then, by the classification of groups of order p^4 , $H/Z(K)$ is of maximal class. It follows that $K'Z(K)/Z(K) = H_3Z(K)/Z(K)$. Hence $[a_1, b] \in H_3Z(K)$ and $[a_1, b, b] \in H_4$. Since $[a_1, b, b] \neq 1$, we have $H_4 \neq 1$ and $c(H) \geq 4$, which is contrary to Theorem 3.4. If $d(H/Z(K)) = 3$, then, by the classification of groups of order p^4 , there exists $d \in H$ such that $H/Z(K) = K/Z(K) \times \langle dZ(K) \rangle$ or $H/Z(K) = K/Z(K) * \langle dZ(K) \rangle$. By calculation, $[d^p, k] = [d, k]^p = 1$ for all $k \in K$. It follows that $d^p \in Z(K)$ and $H/Z(K) = K/Z(K) \times \langle dZ(K) \rangle$. Since $a_2 = [a_1, b] \notin \langle a_1, d \rangle$, by Theorem 3.5, we have $[a_1, d] = 1$. The same reason gives that $[b, d] = 1$. Hence $d \in Z(H)$. In this case $\langle a_2d, b \rangle$ is neither abelian nor normal, a contradiction.

Case 2: $K = \langle a_1, b \rangle$ is a group of Type (18)–(21) in lemma 2.4.

Let $H \leq G$ such that $K < H$. By Theorem 3.5, $H' \leq \langle c, a \rangle \cap \langle c, b \rangle = \langle c, a^p, b^p \rangle$. It follows that $H' = K'$ and $H_3 = K_3 = \langle a^p, b^p \rangle$. By the classification of groups of order p^4 , there exists $d \in H \setminus K$ such that $[a, d] \equiv [b, d] \equiv 1 \pmod{K_3}$. By calculation, $[a, d^p] = [a, d]^p = 1$ and $[b, d^p] = [b, d]^p = 1$. It follows that $d^p \in Z(K) = K_3$. Since $c \notin \langle a, d \rangle$, by Theorem 3.5, we have $[a, d] = 1$. The same reason gives that $[ac, d] = 1$. In this case $\langle a, cd \rangle$ is neither abelian nor normal, a contradiction. \square

Theorem 3.8. *Suppose that G is a finite metahamiltonian p -group having an elementary abelian derived group. If $c(G) = 3$, then G is an \mathcal{A}_2 -group.*

Proof Assume the contrary and G is a counterexample with minimal order. Then $c(G) = 3$ and $G \in \mathcal{A}_3$.

We claim that G does not satisfy the 2-Engel condition. If not, then, by Theorem 2.6, G is a 3-group. In this case, there exist $x, y, z \in G$ such that $[x, y, z] \neq 1$. Since G has minimal order, we have $G = \langle x, y, z \rangle$ and $[x, y, z]^3 = [x^3, y, z] = 1$. Since $[x, yz, yz] = 1$, by calculation, we get $[x, y, z] = [z, x, y]$. Similar reasons give that $[x, y, z] = [y, z, x] = [z, x, y]$. Let $[x, y] = c, [y, z] = a, [z, x] = b, [x, y, z] = [y, z, x] = [z, x, y] = d$. Then $G' = \langle a, b, c, d \rangle$. Since $[b, y] = d \neq 1$, we have $\langle b, y \rangle \trianglelefteq G$. It follows that $c = [x, y] \in \langle b, y \rangle$. Since $[c, b] = [c, y] = 1$, we may assume that $c = y^{3t}d^w$. Hence $d = [c, z] = [y^{3t}d^w, z] = [y^{3t}, z] = 1$, a contradiction.

Since G does not satisfy the 2-Engel condition, there exist $x, y \in G$ such that $[x, y, y] \neq 1$. Since G has minimal order, we have $G = \langle x, y \rangle$, $[x, y, y]^p = 1$ and $[x, y, x]^p = 1$. Let $[x, y] = c, [c, y] = b$ and $[c, x] = a$. Then $G_3 = \langle b, a \rangle$ and $G' = \langle c, G_3 \rangle$. If $[c, x] \in \langle b \rangle$, then, by suitable replacement, we may assume that $[c, x] = a = 1$. Hence we may assume that $\langle a \rangle \cap \langle b \rangle = 1$.

The maximal subgroups of G are $M = \langle x^i y, \Phi(G) \rangle$ and $K = \langle x, \Phi(G) \rangle$, where $i = 0, 1, \dots, p$. It is easy to see that $\Phi(G) = \langle x^p, y^p, c, a, b \rangle$ is abelian. Since $[c, x^i y] = a^i b \neq 1$, by Lemma 2.5 (2), we have that $N = \langle c, x^i y \rangle \in \mathcal{A}_1$. By Theorem 3.5, $G' \leq N$. Since $[cx^p, x^i y] = ba^{i+i(\frac{p}{2})} \neq 1$, by Lemma 2.5 (2), we have $\langle cx^p, x^i y \rangle \in \mathcal{A}_1$. By Lemma 3.3, $\langle cx^p, x^i y \rangle = \langle x^i y \rangle^G = N$. It follows that $x^p \in N$. Since $(x^i y)^p \equiv x^{ip} y^p \pmod{G'}$, we have $x^{ip} y^p \in N$ and hence $y^p \in N$. Thus $\Phi(G) \leq N$ and $M = N \in \mathcal{A}_1$.

If $[c, x] = a \neq 1$, then, by Lemma 2.5 (2), $\langle c, x \rangle \in \mathcal{A}_1$. By Theorem 3.5, $G' \leq L$. Since $[cy^p, x] \neq 1$, by Lemma 2.5 (2), $\langle cy^p, x \rangle \in \mathcal{A}_1$. By Lemma 3.3, $\langle cy^p, x \rangle = \langle x \rangle^G = \langle c, x \rangle$. It follows that $y^p \in \langle c, x \rangle$ and hence $\Phi(G) \leq \langle c, x \rangle$. Thus $K = \langle c, x \rangle \in \mathcal{A}_1$.

If $[c, x] = a = 1$ and $p > 2$, then $[x, y^p] = 1$. Hence $[\Phi(G), x] = 1$ and K is abelian. If $[c, x] = a = 1$ and $p = 2$, then $[x, y^2] = b \neq 1$. By Lemma 2.5 (2), $\langle x, y^2 \rangle \in \mathcal{A}_1$. By 3.5, $G' \leq \langle x, y^2 \rangle$. Hence $K = \langle x, y^2 \rangle \in \mathcal{A}_1$.

By the above argument, all maximal subgroup of G are abelian or minimal non-abelian. Hence $G \in \mathcal{A}_2$, a contradiction. \square

Corollary 3.9. *Suppose that G is a finite metahamiltonian p -group having an elementary abelian derived group. If $c(G) = 3$, then $d(G) = 2$ and p is odd.*

Proof By Theorem 3.8, $G \in \mathcal{A}_2$. Then the results follow from Corollary 2.5 (3). \square

Acknowledgments

We cordially thank Professor Joseph Brennan for his assistance on the exposition and language of the paper.

References

- [1] R. Baer, Situation der Untergruppen und struktur der Gruppen, *S. B. Heidelberg Akad. Mat. Nat.*, **2**(1933), 12–17.

- [2] Y. Berkovich, Groups of Prime Power Order, Vol.1 *Walter de Gruyter, Berlin*, 2008.
- [3] Y. Berkovich and Z. Janko, Structure of finite p -groups with given subgroups, *Contemp. Math.* 402, Amer. Math. Soc., Providence, RI. (2006), 13–93.
- [4] Y. Berkovich and Z. Janko, Groups of Prime Power Order, Vol.2 *Walter de Gruyter, Berlin*, 2008.
- [5] N. Blackburn, On prime power groups with two generators, *Proc. Cambridge Phil. Soc.*, **54**(1957), 327–337.
- [6] R. Dedekind, Über Gruppen, deren sämtliche Teiler Normalteiler sind, *Math. Ann.*, **48**(1897), 548–561.
- [7] S.V. Draganyuk, On the structure of finite primary groups all 2-maximal subgroups of which are abelian (Russian), in Complex analysis, Algebra and topology, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev.(1990), 42–51.
- [8] X.G. Fang and L.J. An, A classification of finite meta-Hamilton p -groups, *in preparation*.
- [9] L.S. Kazarin, On certain classes of finite groups, *Dokl. Akad. Nauk SSSR (Russian)*, **197**(1971), 773–776.
- [10] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, 1967.
- [11] V.T. Nagrebeckii, Invariant coverings of subgroups, *Ural Gos. Univ. Mat. Zap.*, **5**(1966), 91–100.
- [12] V.T. Nagrebeckii, Finite non-nilpotent groups, any non-abelian subgroup of which is normal, *Ural Gos. Univ. Mat. Zap.*, **6**(1967), 80–88.
- [13] V.T. Nagrebeckii, Finite groups in which any non-nilpotent subgroups is invariant, *Ural Gos. Univ. Mat. Zap.*, **7** (1968), 45–49.
- [14] D.S. Passman, Nonnormal subgroups of p -groups, *J. Algebra*, **15**(1970), 352–370.
- [15] L.Rédei, Das schiefe Product in der Gruppentheorie, *Comment.Math.Helvet.*, **20**(1947), 225–267.
- [16] G.M. Romalis and N.F. Sesekin, Metahamiltonian groups, *Ural. Gos. Univ. Mat. Zap.*, **5**(1966), 101–106.
- [17] G.M. Romalis and N.F. Sesekin, Metahamiltonian groups II, *Ural. Gos. Univ. Mat. Zap.*, **6**(1968), 50–52.

- [18] G.M. Romalis and N.F. Sesekin, Metahamiltonian groups III, *Ural. Gos. Univ. Mat. Zap.*, **7**(1969–1970), 195–199.
- [19] V.A. Sheriev, A description of the class of finite p -groups whose 2-maximal subgroups are all abelian II, in Primary groups, *Proc. Sem. Algebraic Systems*, **2**(1970), 54–76.
- [20] M.Y. Xu, L.J. An and Q.H. Zhang, Finite p -groups all of whose non-abelian proper subgroups are generated by two elements, *J. Algebra*, 319(2008), 3603–3620.
- [21] Q.H. Zhang, X.J. Sun, L.J. An and M.Y. Xu, Finite p -groups all of whose subgroups of index p^2 are abelian, *Algebra Colloquium*, **15:1**(2008), 167–180.